Math 10B with Professor Stankova
Worksheet, Discussion #15; Thursday, 3/8/2018
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1 Estimators and Confidence Intervals

1.1 Concepts

1. Often times, we are not given the distribution or parameters of the distribution (but we know what kind of distribution it is), and we want to figure out what the parameters are. One example is if you are given a biased coin and you want to figure out how biased it is (how likely flipping heads/tails is).

   The **estimator for the mean** is the **sample mean** which is given as

$$\hat{\mu} = \bar{x} = \frac{1}{n}\sum_{k=1}^{n} x_k.$$

   The **biased standard deviation estimator** is given by

$$x_* = \sqrt{\frac{1}{n}\sum_{k=1}^{n}(x_k - \bar{x})^2}.$$

   The **unbiased standard deviation** or **sample standard deviation** is given by

$$s = \sqrt{\frac{1}{n-1}\sum_{k=1}^{n}(x_k - \bar{x})^2}.$$

   Given estimators for the mean and standard deviation (or the sample mean and sample standard deviation) $\hat{\mu}, \hat{\sigma}$ respectively, the 95% **confidence interval** for the expected value μ is

$$(\hat{\mu} - 2\hat{\sigma}/\sqrt{n}, \hat{\mu} + 2\hat{\sigma}/\sqrt{n}).$$

   You say that you are 95% confident that μ is in that interval.

1.2 Examples

2. I have a loaded die and I think that it is more likely to be a 1 than normal. Suppose I roll it 100 times and get 1 25 times. What is the 95% confidence interval for p, the probability of getting a 1?

Solution: The estimator for p is $\frac{25/100}{=}\frac{1}{4}$. Given this, the estimator for the standard deviation is $\hat{\sigma} = \sqrt{\hat{p}(1-\hat{p})} = \sqrt{(1/4)(3/4)} = \frac{\sqrt{3}}{4}$. So, the 95% confidence interval is $(\hat{\mu} - 2\hat{\sigma}/\sqrt{n}, \hat{\mu} + 2\hat{\sigma}/\sqrt{n})$, where $n = 100$ to get $(1/4 - \sqrt{3}/20, 1/4 + \sqrt{3}/20)$.

3. Suppose that the amount of lightning strikes at Berkeley per thunderstorm is Poisson distributed. In the past 10 storms, I observe $2, 3, 5, 1, 3, 7, 9, 2, 0,$ and 1 lightning strikes. What is the 95% confidence interval for λ?

Solution: The estimator for λ is the sample mean which is $\frac{2+3+5+1+3+7+9+2+0+1}{10} = 3.3$. Then, the estimator for the sample standard deviation is $\sqrt{\hat{\lambda}} = \sqrt{3.3}$. We had $n = 10$ trials and hence the interval is

$$(3.3 - 2\sqrt{3.3}/\sqrt{10}, 3.3 + 2\sqrt{3.3}/\sqrt{10}) = \left(3.3 - \frac{\sqrt{33}}{5}, 3.3 + \frac{\sqrt{33}}{5}\right).$$

1.3 Problems

4. True **FALSE** A smaller 95% confidence interval means that we are less sure about what the mean μ could be.

Solution: If we have a smaller interval, that means that we are actually more sure about what it could be and we think we know it more precisely.

5. Suppose that you measure the heights of everyone in a 400 person class to estimate the average height of a Berkeley student. Suppose that the sample mean is 65 inches with a standard deviation of 10 inches. What is the 95% confidence interval for the average height of a Berkeley student?

Solution: We have that $n = 400, \mu = 65, \sigma = 10$ and hence the 95% confidence interval is $(65 - 2 \cdot 10/\sqrt{400}, 65 + 2 \cdot 10/\sqrt{400}) = (64, 66)$.

6. Assume the standard deviation of student heights is 5 inches. How large of a sample do you need to be 95% confident that the sample mean is within 1 inch of the population mean?

Solution: We want to be between 1 of the population mean and hence our interval radius $2\sigma/\sqrt{n}$ must be equal to 1. Therefore, we have that $2\sigma/\sqrt{n} = 1$ and $\sigma = 5$ so $\sqrt{n} = 10$ and $n = 100$.

7. In a class of 25 students, the time that students spent on the midterm was 40 minutes with a standard deviation of 5 minutes. What is the 95% confidence interval for the average time taken on the midterm?

Solution: We have that $\mu = 40, n = 25, \sigma = 5$. Plugging it into our formula gives $(40 - 2 \cdot 5/\sqrt{25}, 40 + 2 \cdot 5/\sqrt{25}) = (38, 42)$.

8. In 2012, you interview 10,000 people and ask who they voted for. Out of these people, 51% of people voted for Obama. Are you 95% sure that a majority of people in America support Obama? (Hint: Formulate asking someone who they voted for as a Bernoulli trial)

Solution: Suppose that we say that voting for Obama is a success. Let p be the probability of success. Then this problem is if we do 10,000 trials and get 51% success. So the estimator for p is 0.51 and the estimator for the sample standard deviation is $\sqrt{p(1-p)} = \sqrt{0.51(0.49)} \approx 0.5$. Then, we are 95% confident that the percentage of people support Obama is

$$\left(p - \frac{2\sqrt{p(1-p)}}{\sqrt{n}}, p + \frac{2\sqrt{p(1-p)}}{\sqrt{n}}\right) = \left(0.51 - \frac{1}{\sqrt{10,000}}, 0.51 + \frac{1}{\sqrt{10,000}}\right)$$

$$= (0.51 - 0.01, 0.51 + 0.01) = (0.5, 0.52).$$

So, we are 95% confident that the percentage of people supporting Obama is between 50% and 52%, which means that we are 95% sure that a majority of people support Obama.

1.4 Extra Problems

9. I have a loaded die and I think that it is more likely to be a 1 than normal. Suppose I roll it 100 times and get 1 50 times. What is the 95% confidence interval for p, the probability of getting a 1?

Solution: The estimator for p is $\frac{50}{100} = \frac{1}{2}$. Given this, the estimator for the standard deviation is $\hat{\sigma} = \sqrt{\hat{p}(1-\hat{p})} = \sqrt{(1/2)(1/2)} = \frac{1}{2}$. So, the 95% confidence interval is $(\hat{\mu} - 2\hat{\sigma}/\sqrt{n}, \hat{\mu} + 2\hat{\sigma}/\sqrt{n})$, where $n = 100$ to get $(1/2 - 1/10, 1/2 + 1/10)$.

10. Suppose that the amount of lightning strikes at Berkeley per thunderstorm is Poisson distributed. In the past 9 storms, I observe $2, 6, 1, 1, 1, 3, 4, 2, 2,$ and 6 lightning strikes. What is the 95% confidence interval for λ?

> **Solution:** The estimator for λ is the sample mean which is $\frac{2+6+1+1+3+4+2+2+6}{9} = 3$. Then, the estimator for the sample standard deviation is $\sqrt{\hat{\lambda}} = \sqrt{3}$. We had $n = 9$ trials and hence the interval is
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> $$(3 - 2\sqrt{3}/\sqrt{9}, 3.3 + 2\sqrt{3}/\sqrt{9}) = \left(3 - \frac{2\sqrt{3}}{3}, 3 + \frac{2\sqrt{3}}{3}\right).$$

11. Suppose that you measure the heights of everyone in a 100 person class to estimate the average height of a Berkeley student. Suppose that the sample mean is 68 inches with a standard deviation of 10 inches. What is the 95% confidence interval for the average height of a Berkeley student?

> **Solution:** We have that $n = 100, \mu = 68, \sigma = 10$ and hence the 95% confidence interval is $(68 - 2 \cdot 10/\sqrt{100}, 68 + 2 \cdot 10/\sqrt{100}) = (66, 70)$.

12. Assume the standard deviation of student heights is 8 inches. How large of a sample do you need to be 95% confident that the sample mean is within 1 inch of the population mean?

> **Solution:** We want to be between 1 of the population mean and hence our interval radius $2\sigma/\sqrt{n}$ must be equal to 1. Therefore, we have that $2\sigma/\sqrt{n} = 1$ and $\sigma = 8$ so $\sqrt{n} = 16$ and $n = 256$.

13. In a class of 36 students, the time that students spent on the midterm was 45 minutes with a standard deviation of 10 minutes. What is the 95% confidence interval for the average time taken on the midterm?

> **Solution:** We have that $\mu = 45, n = 36, \sigma = 10$. Plugging it into our formula gives $(45 - 2 \cdot 10/\sqrt{36}, 45 + 2 \cdot 10/\sqrt{36}) = (45 - 10/3, 45 + 10/3)$.

14. In 2016, you interview 100 people and ask who they voted for. Out of these people, 54% of people voted for Hillary. Are you 95% sure that a majority of people in America support Hillary?

Solution: Suppose that we say that voting for Hillary is a success. Let p be the probability of success. Then this problem is if we do 100 trials and get 54% success. So the estimator for p is 0.54 and the estimator for the sample standard deviation is $\sqrt{p(1-p)} = \sqrt{0.54(0.46)} \approx 0.5$. Then, we are 95% confident that the percentage of people support Hillary is

$$\left(p - \frac{2\sqrt{p(1-p)}}{\sqrt{n}}, p + \frac{2\sqrt{p(1-p)}}{\sqrt{n}}\right) = \left(0.54 - \frac{1}{\sqrt{100}}, 0.54 + \frac{1}{\sqrt{100}}\right)$$

$$= (0.54 - 0.1, 0.54 + 0.1) = (0.44, 0.64).$$

So, we are 95% confident that the percentage of people supporting Hillary is between 44% and 64%, which means that we are not 95% sure that a majority of people support Hillary.

2 Hypothesis Testing

2.1 Concepts

15. In general, statistics does not allow you to prove anything is true, but instead allows you to show that things are probably false. So when we do hypothesis testing, the **null hypothesis** H_0 is something that we want to show is false and the **alternative hypothesis** H_1 is something that you want to show is true. For example, to show that a drug cures cancer, the null hypothesis would be that the drug does nothing and the alternative hypothesis would be that the drug does help cure cancer.

    A **type 1 error** is rejecting a true null which means that in our example, saying a drug cures cancer when it doesn't. A **type 2 error** is failing to reject a false null which means in our case as saying that the drug doesn't do anything when it does. The **significance level** is the probability of making a type 1 error. The **power** is 1 minus the probability of making a type 2 error.

    You use a χ^2 test to determine if a distribution is how you expect it to be. Suppose that you expect it to be distributed with a different values and for each of these values, you expect to get outcome k m_k times but actually get it n_k times. Then you compare the statistic

$$r = \sum_{k=1}^{a} \frac{(n_k - m_k)^2}{m_k}$$

    with the $\chi^2(a-1)$ distribution.

2.2 Examples

16. Chip bags say that they have 14 ounces of chips inside with a standard deviation of 0.5 ounces. You weigh 100 bags and get an average of 13.8 ounces. What can you say with significance level $\alpha = 0.05$?

> **Solution:** The null hypothesis is that the chips have a weight of 14 ounces. The central limit theorem tells us that the standard deviation of the sample is $0.5/\sqrt{n} = 0.5/\sqrt{100} = 0.05$. We calculate the z score as $z(\frac{|a-\mu|}{\sigma/\sqrt{n}}) = z(\frac{|13.8-14|}{0.05}) = z(4) < \alpha/2$. Thus, we can say that the chip bag makers are lying.

17. In a skittle bag, you get 11 red skittles, 12 blue, 5 green, 10 yellow, and 13 orange skittles. Is it possible that the colors are evenly distributed with a significance level of $\alpha = 0.05$?

> **Solution:** In 50 skittles, we expect to get 10 of each. Following the formula, our statistic is:
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> $$\frac{(11-10)^2}{10} + \frac{(12-10)^2}{10} + \frac{(5-10)^2}{10} + \frac{(10-10)^2}{10} + \frac{(13-10)^2}{10}$$
>
> $$= \frac{1+4+25+0+9}{10} = 3.9.$$
>
> There are 5 options so we have $5-1 = 4$ degrees of freedom. For 4 degrees of freedom and $\alpha = 0.05$, our critical value is 9.488. Since $3.9 < 9.488$, we cannot reject the null hypothesis that the colors are evenly distributed.

2.3 Problems

18. **TRUE** False The null hypothesis is something we want to be false.

19. True **FALSE** If we get a value that is not smaller than α, then we have shown that the null hypothesis is true.

> **Solution:** We simply do not have enough evidence to show that it is false, not proven that it is true.

20. True **FALSE** We want our test to have a high significance level and high power.

Solution: The significance level is the probability of making a type 1 error so we want that low and the power is 1 minus the probability of making a type 2 error so we want that large. So small significance level and high power.

21. **TRUE** False A type-2 error made by a road patrol may result in letting drunken drivers continue driving.

Solution: A type-2 error is when you fail to reject the null hypothesis. We are asking if someone is drunk driving so the null hypothesis, the thing we want to disprove is that he is not drunk driving. So we fail to reject this so that means that they are drunk driving but we do not think so.

22. You flip a coin 100 times and get 55 heads. Can you say that it is biased towards heads? (use $\alpha = 0.05$)

Solution: The null hypothesis is that the coin is unbiased and hence $p = 0.5$. The standard deviation is $\sigma = \sqrt{p(1-p)} = 0.5$. Thus, the central limit theorem tells us that the percentage of coin flips we get is approximately normally distributed with standard deviation $0.5/\sqrt{n} = 0.5/\sqrt{100} = 0.05$. We got $55/100 = 0.55$ percent of heads. Calculating the z score of 0.55 is $z(|0.55 - 0.5|/\sigma) = z(1) > \alpha$ and hence we cannot reject the null hypothesis.

23. An infomercial claims that a miracle drug will cause you to grow all your hair back. There are 25 brave participants and surprisingly 10 people regrew their hair. If normally 10% of people regrow their hair, can you say that this drug worked?

Solution: We would expect that 10% of people will regrow their hair with standard deviation $\sigma = \sqrt{p(1-p)} = \sqrt{0.1(0.9)} = 0.3$. The central limit theorem says that with a sample of 25% people, we expect that 10% of people regrow their hair with a standard deviation of $\sigma/\sqrt{n} = 0.3/\sqrt{25} = 0.06$. There are $10/25 = 40\%$ who regrew their hair. The z score is $z(|0.4 - 0.1|/0.06) = z(5) < \alpha$. Therefore, we can reject the null hypothesis and say that this drug does help you grow your hair back.

24. You take 400 cards and get 100 spades, 105 hearts, 107 diamonds, and 88 clubs. Can you say that the suits are not evenly distributed with $\alpha = 0.05$?

Solution: If they are evenly distributed, we would expect 100 of each. Calculating the statistic gives

$$r = \frac{(100-100)^2}{100} + \frac{(105-100)^2}{100} + \frac{(107-100)^2}{100} + \frac{(88-100)^2}{100}$$
$$= \frac{25+49+144}{100} = 2.18.$$

For $4-1=3$ degrees of freedom, our critical value for α is 7.815 and since $2.18 < 7.815$, we cannot reject the null hypothesis.

25. You expect to get a distribution of brown eyes brown hair to brown eyes blond hair to blue eyes brown hair to blue eyes blond hair as $9:3:3:1$. When looking around in class, you get a distribution of $61:19:11:9$ after looking at 100 people. Is this distribution accurate (use $\alpha = 0.05$)?

Solution: If it were accurately distributed, we would expect $56.25 : 18.75 : 18.75 : 6.25$. So computing the statistic gives

$$r = \frac{(61-56.25)^2}{56.25} + \frac{(19-18.75)^2}{18.75} + \frac{(11-18.75)^2}{18.75} + \frac{(9-6.25)^2}{9.25}$$
$$\approx 4.43.$$

Since $4.43 < 7.815$ which is the critical value for $4-1=3$ degrees of freedom and $\alpha = 0.05$, we cannot reject the null hypothesis.

2.4 Extra Problems

26. Chip bags say that they have 2 ounces of chips inside with a standard deviation of 0.05 ounces. You weigh 100 bags and get an average of 2.01 ounces. What can you say with significance level $\alpha = 0.05$?

Solution: The null hypothesis is that the chips have a weight of 2 ounces. The central limit theorem tells us that the standard deviation of the sample is $0.05/\sqrt{n} = 0.05/\sqrt{100} = 0.005$. We calculate the z score as $z(\frac{|a-\mu|}{\sigma/\sqrt{n}}) = z(\frac{|2.01-2|}{0.005}) = z(2) < \alpha/2$. Thus, we can say that the chip bag makers are lying.

27. In a skittle bag, you get 15 red skittles, 8 blue, 5 green, 7 yellow, and 15 orange skittles. Is it possible that the colors are evenly distributed with a significance level of $\alpha = 0.05$?

Solution: In 50 skittles, we expect to get 10 of each. Following the formula, our statistic is:

$$\frac{(15-10)^2}{10}+\frac{(8-10)^2}{10}+\frac{(5-10)^2}{10}+\frac{(7-10)^2}{10}+\frac{(15-10)^2}{10}$$
$$=\frac{25+4+25+9+25}{10}=8.8.$$

There are 5 options so we have $5-1=4$ degrees of freedom. For 4 degrees of freedom and $\alpha=0.05$, our critical value is 9.488. Since $8.8<9.488$, we cannot reject the null hypothesis that the colors are evenly distributed.

28. You flip a coin 100 times and get 38 heads. Can you say that it is biased towards tails? (use $\alpha=0.05$)

Solution: The null hypothesis is that the coin is unbiased and hence $p=0.5$. The standard deviation is $\sigma=\sqrt{p(1-p)}=0.5$. Thus, the central limit theorem tells us that the percentage of coin flips we get is approximately normally distributed with standard deviation $0.5/\sqrt{n}=0.5/\sqrt{100}=0.05$. We got $38/100=0.38$ percent of heads. Calculating the z score of 0.38 is $z(|0.38-0.5|/\sigma)=z(2.4)<\alpha$ and hence we can reject the null hypothesis and say that it is biased towards tails.

29. An infomercial claims that a miracle drug will cause you to grow all your hair back. There are 25 brave participants and surprisingly 7 people regrew their hair. If normally 10% of people regrow their hair, can you say that this drug worked?

Solution: We would expect that 10% of people will regrow their hair with standard deviation $\sigma=\sqrt{p(1-p)}=\sqrt{0.1(0.9)}=0.3$. The central limit theorem says that with a sample of 25% people, we expect that 10% of people regrow their hair with a standard deviation of $\sigma/\sqrt{n}=0.3/\sqrt{25}=0.06$. There are $7/25=28\%$ who regrew their hair. The z score is $z(|0.28-0.1|/0.06)=z(3)<\alpha$. Therefore, we can reject the null hypothesis and say that this drug does help you grow your hair back.

30. You take 400 cards and get 95 spades, 110 hearts, 109 diamonds, and 86 clubs. Can you say that the suits are not evenly distributed with $\alpha=0.05$?

Solution: If they are evenly distributed, we would expect 100 of each. Calculating the statistic gives

$$r=\frac{(95-100)^2}{100}+\frac{(110-100)^2}{100}+\frac{(109-100)^2}{100}+\frac{(86-100)^2}{100}$$

$$= \frac{25 + 100 + 81 + 196}{100} = 4.02.$$

For $4 - 1 = 3$ degrees of freedom, our critical value for α is 7.815 and since $4.02 < 7.815$, we cannot reject the null hypothesis.